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On Characterizations of the Space of p-Semi-Integral Multilinear Mappings

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Abstract. In this paper we consider the ideal of p-semi-integral n-linear mappings, which is a natural multilinear extension of the ideal of p-summing linear operators. The space of p-semi-integral multilinear mappings is characterized by means of a suitable tensor norm up to an isometric isomorphism. In this connection we also consider tensor products of linear operators and multilinear mappings of finite type.

Keywords: p-semi integral multilinear mappings, tensor product of Banach spaces

MSC 2000 classification: primary 46G25, secondary 46A32

Introduction

Semi-integral multilinear mappings between Banach spaces were introduced by R. Alencar and M. Matos [1] as a natural multilinear extension of the classical ideal of absolutely summing linear operators. The extension of this notion to p-semi-integral multilinear mappings, $1 \leq p < +\infty$ is immediate [see [2, 11]]. It is shown in [11] that the class of p-semi-integral multilinear mappings has many good properties, e.g. the ideal property [11, Proposição 5.1.11], inclusion property [11, Proposição 5.1.9], etc. [see also [2]]. Also it follows from a result of V. Dimant [4] that p-semi integral multilinear mappings have good properties with respect to the Aron-Berner extensions. As well, R. Alencar and M. Matos in [1] show that every multilinear vector-valued Pietsch-integral mapping is semi integral. We refer to [2] and [11] for the relation between p-semi-integral multilinear mappings and other classes of p-summing multilinear mappings, such as dominated, multiple (or, fully), strongly and absolutely summing mappings.

The aim of this paper is to obtain characterizations of the space $\mathcal{L}_{si,p}(E_1,\ldots,E_n;F)$ of p-semi-integral n-linear mappings from $E_1\times\cdots\times E_n$ to F. In Section 2 we introduce a reasonable crossnorm $\widetilde{\sigma}_p$ such that the space $\mathcal{L}_{si,p}(E_1,\ldots,E_n;F')$ of p-semi-integral n-linear mappings is isometric to the dual

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of $E_1 \otimes \cdots \otimes E_n \otimes F$ endowed with $\widetilde{\sigma}_p$. A corresponding reasonable crossnorm σ_p for scalar-valued p-semi-integral mappings is also studied. In Section 3 we study the continuity of the tensor product of linear operators with respect to the norm $\widetilde{\sigma}_p$ (and σ_p). Finally, in Section 4 we consider the norm $\widetilde{\sigma}_p$ (and σ_p) in connection with spaces of multilinear mappings of finite type. Stronger representation results are obtained for multilinear mappings of finite type on reflexive spaces.

The symbols $E, E_1, \ldots, E_n, G_1, \ldots, G_n, F, F_0$ represent (real or complex) Banach spaces, E' denotes the topological dual of E, $\mathbb{K}$ represents the scalar field and $\mathbb{N}$ represents the set of all positive integers. Given a natural number $n \geq 2$, the Banach space of all continuous n-linear mappings from $E_1 \times \cdots \times E_n$ into F endowed with the sup norm will be denoted by $\mathcal{L}(E_1, \ldots, E_n;F)$ ($\mathcal{L}(E_1, \ldots, E_n)$ if $F = \mathbb{K}$). For $p \geq 1$, $l_p(E)$ denotes the linear space of absolutely p-summable sequences $(x_j)_{j=1}^{\infty}$ in E with the norm $\|(x_j)_{j=1}^{\infty}\|_p = \left(\sum_{j=1}^{\infty} \|x_j\|^p\right)^{\frac{1}{p}} < \infty$. Also, $l_p^w(E)$ denotes the linear space of the sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in l_p$ for every $\varphi \in E'$. The expression

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{E'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_p$$

defines a norm on $l_p^w(E)$. If $p = \infty$ we are restricted to the case of bounded sequences and in $l_\infty(E)$ we use the sup norm. The symbol $E_1 \otimes \cdots \otimes E_n$ denotes the algebraic tensor product of the Banach spaces $E_1, \ldots, E_n$.

Let $p \geq 1$. An n-linear mapping $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is *p-semi-integral* ($T \in \mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$) if there exist $C \geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra of $B_{E_1'} \times \cdots \times B_{E_n'}$ endowed with the product of the weak star topologies $\sigma(E_l', E_l)$, $l = 1, \ldots, n$, such that

$$\|T(x_1, \ldots, x_n)\| \leq C \left(\int_{B_{E_1'} \times \cdots \times B_{E_n'}} |\varphi_1(x_1) \cdots \varphi_n(x_n)|^p d\mu(\varphi_1, \ldots, \varphi_n) \right)^{1/p}$$

for every $x_j \in E_j$ and $j = 1, \ldots, n$. The infimum of the constants C working in the inequality defines a norm $\|\cdot\|_{si,p}$ on $\mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$.

1 p-Semi-Integral Mappings and Tensor Products of Banach Spaces

The following characterization of p-semi-integral mappings, which was proved in [11] [see also [2]] will be important in this paper:

1 Theorem. *[11], [2] Let $E_1, \ldots, E_n$ and F be Banach spaces and let $p \geq 1$. Then, $T \in \mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$ if and only if there exists $C \geq 0$ such that*

$$\left(\sum_{j=1}^{m} \|T(x_{1,j}, \ldots, x_{n,j})\|^p \right)^{1/p} \leq C \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p} \tag{1}$$

for every $m \in \mathbb{N}$, $x_{l,j} \in E_l$ with $l = 1, \ldots, n$ and $j = 1, \ldots, m$. Moreover, the infimum of the C in (1) is $\|T\|_{si,p}$.

A standard argument shows that $\mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$ is complete with respect to the norm $\|\cdot\|_{si,p}$. Next we introduce a reasonable crossnorm [see [14, p. 127]] on $E_1 \otimes \cdots \otimes E_n \otimes F$ so that the topological dual of the resulting space is isometric to $(\mathcal{L}_{si,p}(E_1, \ldots, E_n; F'), \|\cdot\|_{si,p})$.

2 Proposition. *Let $E_1, \ldots, E_n$ and F be Banach spaces and let $p \geq 1$. Let*

$$\widetilde{\sigma}_p(u) := \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p} \|(b_j)_{j=1}^m\|_\infty$$

where the infimum is taken over all representations of $u \in E_1 \otimes \cdots \otimes E_n \otimes F$ in the form

$$u = \sum_{j=1}^{m} \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j$$

with $m \in \mathbb{N}$, $x_{l,j} \in E_l$, $l = 1, \ldots, n$, $\lambda_j \in \mathbb{K}$, $b_j \in F$, $j = 1, \ldots, m$, and $q \geq 1$ with $1/p + 1/q = 1$.

Then the function $\widetilde{\sigma}_p$ is a reasonable crossnorm on $E_1 \otimes \cdots \otimes E_n \otimes F$.

For the proof we will need the following lemma.

3 Lemma. *Given $u \in E_1 \otimes \cdots \otimes E_n \otimes F$, for any $\delta > 0$ we can find a representation of u of the form*

$$u = \sum_{j=1}^{m} \alpha_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes a_j,$$

such that

$$\| (\alpha_j)_{j=1}^m \|_q \leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q},$$

$$\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} | \varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j}) |^p \leq (1+\delta)\widetilde{\sigma}_p(u),$$

$$\| (a_j)_{j=1}^m \|_\infty = 1.$$

PROOF. Let us take a constant $\delta > 0$. It is clear, by the definition of $\widetilde{\sigma}_p$, that we can choose a representation of u of the form

$$u = \sum_{j=1}^m \alpha_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes a_j,$$

such that

$$\widetilde{\sigma}_p(u) \leq \| (\alpha_j)_{j=1}^m \|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m \mid \varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j}) \mid^p \right)^{1/p} \| (a_j)_{j=1}^m \|_\infty \tag{*}$$

$$\leq (1+\delta)\widetilde{\sigma}_p(u) = [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q}[(1+\delta)\widetilde{\sigma}_p(u)]^{1/p}.$$

Thus as a first step we can rearrange the representation of u by multiplying and dividing $\| (a_j)_{j=1}^m \|_\infty$ with a suitable constant $c > 0$ so that $\| (a_j^*)_{j=1}^m \|_\infty := \| (ca_j)_{j=1}^m \|_\infty = 1$, and $\| (\alpha_j^*)_{j=1}^m \|_q := \| (\frac{1}{c}\alpha_j)_{j=1}^m \|_q$. Observe that the representation $u = \sum_{j=1}^m \alpha_j^* x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes a_j^*$ satisfies $(*)$ with

$$\| (\alpha_j^*)_{j=1}^m \|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m \mid \varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j}) \mid^p \right)^{1/p} \leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q}[(1+\delta)\widetilde{\sigma}_p(u)]^{1/p}.$$

Now as a second step, for this representation of u, for example, if

$$\left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m \mid \varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j}) \mid^p \right)^{1/p} > [(1+\delta)\widetilde{\sigma}_p(u)]^{1/p} \tag{**}$$

again we can choose a suitable constant $C > 0$ so that

$$\left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m \mid \varphi_1(Cx_{1,j}) \cdots \varphi_n(x_{n,j}) \mid^p \right)^{1/p} = [(1+\delta)\widetilde{\sigma}_p(u)]^{1/p}.$$

Hence, we have that

$$\| (\alpha_j^*)_{j=1}^m \|_q \frac{1}{C} \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^m | \varphi_1(Cx_{1,j}) \cdots \varphi_n(x_{n,j}) |^p \right)^{1/p} \| (a_j^*)_{j=1}^m \|_\infty$$
$$\leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q} [(1+\delta)\widetilde{\sigma}_p(u)]^{1/p}$$

and this will imply that $\| (\alpha_j^*)_{j=1}^m \|_q \frac{1}{C} \leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q}$. Now taking $\| (\alpha_j^{**})_{j=1}^m \|_q = \| (\frac{1}{C}\alpha_j^*)_{j=1}^m \|_q$ and $x_{1,j}^* = Cx_{1,j}$, $j = 1, \ldots, m$ we obtain a representation of u of the form $u = \sum_{j=1}^m \alpha_j^{**} x_{1,j}^* \otimes \cdots \otimes x_{n,j} \otimes a_j^*$ satisfying $(*)$ and conditions

$$\| (\alpha_j^{**})_{j=1}^m \|_q \leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q},$$

$$\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^m | \varphi_1(x_{1,j}^*) \cdots \varphi_n(x_{n,j}) |^p \leq (1+\delta)\widetilde{\sigma}_p(u),$$

$$\| (a_j^*)_{j=1}^m \|_\infty = 1.$$

Note that, in the second step above, if, instead of (**), it would be

$$\| (\alpha_j^*)_{j=1}^m \|_q > [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q}, \tag{***}$$

then we would proceed completely in a similar way to obtain a suitable representation of u satisfying (*) and the above conditions. Note also that, as a consequence of the inequality (*), it cannot happen (**) and (***) simultaneously. $\boxed{QED}$

Proof of Proposition 2. First we show that $\widetilde{\sigma}_p(u) = 0$ implies $u = 0$. Suppose that $\widetilde{\sigma}_p(u) = 0$. Then, for every $\epsilon > 0$, there is a representation $\sum_{j=1}^m \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j$ of u such that

$$\| (\lambda_j)_{j=1}^m \|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^m | \varphi_1(x_{1,j}) \ldots \varphi_n(x_{n,j}) |^p \right)^{1/p} \| (b_j)_{j=1}^m \|_\infty < \epsilon.$$

Hence it follows from the Hölder's inequality that

$$\sup_{\substack{\varphi_l \in B_{E_l'}, \varphi \in B_{F'} \\ l=1,\dots,n}} \left| \varphi_1 \times \cdots \times \varphi_n \times \varphi(\sum_{j=1}^m \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j) \right|$$

$$= \sup_{\substack{\varphi_l \in B_{E_l'}, \varphi \in B_{F'} \\ l=1,\dots,n}} \left| \sum_{j=1}^m \varphi_1(\lambda_j x_{1,j}) \cdots \varphi_n(x_{n,j}) \varphi(b_j) \right|$$

$$\leq \| (b_j)_{j=1}^m) \|_\infty \| (\lambda_j)_{j=1}^m) \|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m \mid \varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j}) \mid^p \right)^{1/p} < \epsilon.$$

Thus we have that

$$\left| \sum_{j=1}^m \varphi_1(\lambda_j x_{1,j}) \cdots \varphi_n(x_{n,j}) \varphi(b_j) \right| < \epsilon \| \varphi_1 \| \cdots \| \varphi_n \| \| \varphi \|,$$

for every $\varphi_l \in E_l'$, $l = 1, \dots, n$ and $\varphi \in F'$.

Since the value of the sum $\left| \varphi_1 \times \cdots \times \varphi_n \times \varphi(\sum_{j=1}^m \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j) \right|$ is independent of the representation of u, it follows that

$$\sum_{j=1}^m \varphi_1(\lambda_j x_{1,j}) \cdots \varphi_n(x_{n,j}) \varphi(b_j) = 0,$$

for every $\varphi_l \in E_l'$, $l = 1, \dots, n$, $\varphi \in F'$.

Hence, since $E_1', \dots, E_n'$ and F' are separating subsets of the respective algebraic duals, by the multilinear version of [14, Proposition 1.2] it follows that $u = 0$.

To prove the triangular inequality, take $u, v \in E_1 \otimes \cdots \otimes E_n \otimes F$. For any $\delta > 0$, by Lemma 3 we can find representations

$$u = \sum_{j=1}^m \alpha_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes a_j \qquad \text{and} \qquad v = \sum_{j=1}^m \beta_j y_{1,j} \otimes \cdots \otimes y_{n,j} \otimes b_j$$

such that

$$\|(\alpha_j)_{j=1}^m\|_q \leq [(1+\delta)\widetilde{\sigma}_p(u)]^{1/q},$$

$$\|(\beta_j)_{j=1}^m\|_q \le [(1+\delta)\widetilde{\sigma}_p(v)]^{1/q},$$

$$\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\varphi_1(x_{1,j})\cdots\varphi_n(x_{n,j})|^p \le (1+\delta)\widetilde{\sigma}_p(u),$$

$$\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\varphi_1(y_{1,j})\cdots\varphi_n(y_{n,j})|^p \le (1+\delta)\widetilde{\sigma}_p(v),$$

$$\|(a_j)_{j=1}^m\|_\infty = 1 = \|(b_j)_{j=1}^m\|_\infty.$$

Then it follows that

$$\begin{aligned}
\widetilde{\sigma}_p(u+v) &\le \left(\sum_{j=1}^m |\alpha_j|^q + \sum_{j=1}^m |\beta_j|^q\right)^{1/q} \\
&\quad \times \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \left(\sum_{j=1}^m |\varphi_1(x_{1,j})\cdots\varphi_n(x_{n,j})|^p + \sum_{j=1}^m |\varphi_1(y_{1,j})\cdots\varphi_n(y_{n,j})|^p\right)\right)^{1/p} \\
&\le (1+\delta)^{1/q}(\widetilde{\sigma}_p(u)+\widetilde{\sigma}_p(v))^{1/q}(1+\delta)^{1/p}(\widetilde{\sigma}_p(u)+\widetilde{\sigma}_p(v))^{1/p} \\
&= (1+\delta)(\widetilde{\sigma}_p(u)+\widetilde{\sigma}_p(v)),
\end{aligned}$$

which shows the triangular inequality. Hence $\widetilde{\sigma}_p$ is a norm on $E_1 \otimes \cdots \otimes E_n \otimes F$.

It is easily seen that $\widetilde{\sigma}_p(x_1 \otimes \cdots \otimes x_n \otimes b) \le \|x_1\|\cdots\|x_n\| \cdot \|b\|$ for every $x_l \in E_l$, $l = 1,\dots,n$ and $b \in F$. To show that $\|\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \varphi\| \le \|\varphi_1\|\cdots\|\varphi_n\| \cdot \|\varphi\|$ let $\varphi_l \in E_l'$ with $\varphi_l \neq 0$, $l = 1,\dots,n$, let $\varphi \in F'$ with $\varphi \neq 0$, and let $u = \sum_{j=1}^m \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j$. Then by the Hölder's inequality we get

$$\begin{aligned}
|\varphi_1 \otimes \cdots \otimes \varphi_n(u)| &\le \|\varphi\| \|(b_j)_{j=1}^m\|_\infty \|\varphi_1\|\cdots\|\varphi_n\| \|(\lambda_j)_{j=1}^m\|_q \\
&\quad \times \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\varphi_1(x_{1,j})\cdots\varphi_n(x_{n,j})|^p\right)^{1/p}.
\end{aligned}$$

Therefore we obtain that $|\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \varphi(u)| \le \|\varphi_1\|\cdots\|\varphi_n\|\|\varphi\|\widetilde{\sigma}_p(u)$, and we have shown that $\widetilde{\sigma}_p$ is a reasonable crossnorm. $\boxed{QED}$

Note that when $n = 1$, in particular, the norm $\widetilde{\sigma}_p$ is reduced to the Chevet-Saphar norm d_q on $E_1 \otimes F$ [see [14, pg. 135]].

In the previous proposition if we take $F = \mathbb{K}$, then we identify $E_1 \otimes \cdots \otimes E_n \otimes \mathbb{K}$ with $E_1 \otimes \cdots \otimes E_n$, and in this case the corresponding reasonable crossnorm will be denoted by σ_p which is described as follows:

$$\sigma_p(u) := \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p}$$

where the infimum is taken over all representations of $u \in E_1 \otimes \cdots \otimes E_n$ in the form $u = \sum_{j=1}^m \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j}$ with $m \in \mathbb{N}$, $x_{l,j} \in E_l$, $l = 1, \dots, n$, $\lambda_j \in \mathbb{K}$, $j = 1, \dots, m$, and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

4 Remark. (Commutativity and associativity of σ_p) Let E, F and G be Banach spaces. Since the algebraic isomorphisms $E \otimes F = F \otimes E$ and $E \otimes (F \otimes G) = (E \otimes F) \otimes G$ are well known [see, for example, [7, p. 179]] then it follows by the very definition of σ_p that, the normed (resp. Banach) spaces $(E \otimes F, \sigma_p)$ and $(F \otimes E, \sigma_p)$ (resp. $(E \tilde{\otimes} F, \sigma_p)$ and $(F \tilde{\otimes} E, \sigma_p)$) are isometrically isomorphic, and the normed (resp. Banach) spaces $((E \otimes F, \sigma_p) \otimes G, \sigma_p)$ and $(E \otimes (F \otimes G, \sigma_p), \sigma_p)$ (resp. $((E \tilde{\otimes} F, \sigma_p) \tilde{\otimes} G, \sigma_p)$ and $(E \tilde{\otimes} (F \tilde{\otimes} G, \sigma_p), \sigma_p)$) are isometrically isomorphic in the canonical way, where the symbol $\tilde{\otimes}$ denotes the completion of the corresponding normed space.

The above remark assures that the (reasonable) crossnorm σ_p is symmetric, that is, if we interchange the factor spaces the value of the norm does not alter. Although σ_p and $\widetilde{\sigma}_p$ share many properties, let us see that, contrary to the case of σ_p, commutativity and associativity do not hold for $\widetilde{\sigma}_p$: take a tensor u in $E \otimes F$ and consider the infima

$$\inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\varphi \in B_{E'}} \sum_{j=1}^m |\varphi(x_j)|^p \right)^{1/p} \|(y_j)_{j=1}^m\|_\infty \text{ and}$$

$$\inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\phi \in B_{F'}} \sum_{j=1}^m |\phi(y_j)|^p \right)^{1/p} \|(x_j)_{j=1}^m\|_\infty,$$

where the infima are taken over all representations $u = \sum_{j=1}^m \lambda_j x_j \otimes y_j$ with $\lambda_j \in \mathbb{K}$, $x_j \in E$, $y_j \in F$, $j = 1, \dots, m$. The fact that these infima are different in general shows that $\widetilde{\sigma}_p$ is not a symmetric norm. Its non-associativity follows analogously.

5 Remark. Let $E_1, \ldots, E_n$ and F be Banach spaces and let $p \geq 1$.

(a) It follows from the definitions of σ_p and $\widetilde{\sigma}_p$ that $\sigma_p(u) \leq \widetilde{\sigma}_p(u)$ for every $u \in E_1 \otimes \cdots \otimes E_n \otimes F$.

(b) To each tensor $u \in E_1' \otimes \cdots \otimes E_n'$ corresponds a canonical operator $T_u \colon E_1 \times \cdots \times E_n \longrightarrow \mathbb{K}$ given by

$$u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \otimes \cdots \otimes \varphi_{n,j} \mapsto T_u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \times \cdots \times \varphi_{n,j},$$

with $\lambda_j \in \mathbb{K}$, $\varphi_{l,j} \in E_l^{'}$, $l = 1, \ldots, n$, $j = 1, \ldots, m$. By an easy application of Hölder's inequality we see that $\|T_u\| \leq \sigma_p(u)$ for every $u \in E_1' \otimes \cdots \otimes E_n'$.

Below by combining the argument of the proof of [9, Theorem 3.7] with Theorem 1 we prove the following result. This result characterizes the space of p-semi integral mappings as the topological dual of the space of the tensor product $(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)$ up to an isometric isomorphism.

6 Proposition. *Let $E_1, \ldots, E_n$ be Banach spaces. Then, for every Banach space F and $p \geq 1$, the space $(\mathcal{L}_{si,p}(E_1, \ldots, E_n; F^{'}), \|\cdot\|_{si,p})$ is isometrically isomorphic to $(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)^{'}$ through the mapping $T \longrightarrow \phi_T$, where $\phi_T(x_1 \otimes \cdots \otimes x_n \otimes b) = T(x_1, \ldots, x_n)(b)$, for every $x_l \in E_l$, $l = 1, \ldots, n$, and $b \in F$.*

Proof. It is easy to see that the correspondence

$$T \in \mathcal{L}_{si,p}(E_1, \ldots, E_n; F') \longrightarrow \phi_T \in (E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)^{'}$$

defined by

$$\phi_T(x_1 \otimes \cdots \otimes x_n \otimes b) := T(x_1, \ldots, x_n)(b), \quad x_l \in E_l, \quad l = 1, \ldots, n \text{ and } b \in F,$$

is linear and injective. To show the surjectivity let $\phi \in (E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'$ and consider the corresponding n-linear mapping $T_\phi \in \mathcal{L}(E_1, \ldots, E_n; F')$, defined by $T_\phi(x_1, \ldots, x_n)(b) = \phi(x_1 \otimes \cdots \otimes x_n \otimes b)$, for $x_l \in E_l$, $l = 1, \ldots, n$, and $b \in F$. Let us consider $x_{l,j} \in E_l$, $l = 1, \ldots, n$, $j = 1, \ldots, m$. For every $\epsilon > 0$ there are $b_j \in F$, with $\|b_j\| = 1$, $j = 1, \ldots, m$, such that

$$\begin{aligned}\|(T_\phi(x_{1,j}, \ldots, x_{n,j}))_{j=1}^m\|_p^p &= \sum_{j=1}^{m} \|T_\phi(x_{1,j}, \ldots, x_{n,j})\|^p \\ &\leq \epsilon + \sum_{j=1}^{m} |T_\phi(x_{1,j}, \ldots, x_{n,j})(b_j)|^p = (*).\end{aligned}$$

Now we can choose $\lambda_j \in \mathbb{K}$, with $|\lambda_j| = 1$, $j = 1, \ldots, m$, such that

$$\begin{aligned}(*) &= \epsilon + \sum_{j=1}^{m} |\phi(x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j)|^p \\ &= \epsilon + \left| \sum_{j=1}^{m} |\phi(x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j)|^{p-1} \lambda_j \phi(x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j) \right| = (**).\end{aligned}$$

Proceeding from this point, by continuity of ϕ and the Hölder's inequality we get

$$\begin{aligned}(**) \leq & \epsilon + \|\phi\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'} \widetilde{\sigma}_p \\ & \left(\sum_{j=1}^{m} \lambda_j |\phi(x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j)|^{p-1} x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j \right) \\ \leq & \epsilon + \|\phi\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'} \left\| \left(\lambda_j |\phi(x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j)|^{p-1} \right)_{j=1}^{m} \right\|_q \\ & \times \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p} \|(b_j)_{j=1}^{m}\|_\infty \\ = & \epsilon + \|\phi\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'} \left\| (T_\phi(x_{1,j}, \ldots, x_{n,j}))_{j=1}^{m} \right\|_p^{p/q} \\ & \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p}.\end{aligned}$$

Since ϵ is arbitrary and $p - (p/q) = 1$ we obtain

$$\begin{aligned}\|(T_\phi(x_{1,j}, \ldots, x_{1,j}))_{j=1}^{m}\|_p \leq & \|\phi\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'} \\ & \left(\sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p \right)^{1/p},\end{aligned}$$

showing that $\|T_\phi\|_{si,p} \leq \|\phi\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'}$, and therefore
$T_\phi \in (\mathcal{L}_{si,p}(E_1, \ldots, E_n; F'), \| \cdot \|_{si,p})$.

To show the reverse inequality let $T \in \mathcal{L}_{si,p}(E_1, \ldots, E_n; F')$ and consider the linear functional ϕ_T on $E_1 \otimes \cdots \otimes E_n \otimes F$ given by

$$\phi_T(u) = \sum_{j=1}^{m} \lambda_j T(x_{1,j}, \ldots, x_{n,j})(b_j)$$

for $u = \sum_{j=1}^{m} \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes b_j$, where $m \in \mathbb{N}$, $\lambda_j \in \mathbb{K}$, $k = 1, \ldots, n$, $b_j \in F$, $j = 1, \ldots, m$. Hence, by Hölder's inequality and Theorem 1 it follows that

$$|\phi_T(u)|^p \leq \|(\lambda_j)_{j=1}^m\|_q^p \|(b_j)_{j=1}^m\|_\infty^p \|T\|_{si,p}^p \sup_{\substack{\varphi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\varphi_1(x_{1,j}) \cdots \varphi_n(x_{n,j})|^p.$$

Thus $|\phi_T(u)| \leq \|T\|_{si,p} \widetilde{\sigma}_p(u)$, showing that ϕ_T is $\widetilde{\sigma}_p$-continuous with $\|\phi_T\|_{(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)'} \leq \|T\|_{si,p}$. $\boxed{QED}$

Making $F = \mathbb{K}$, in the previous Proposition we obtain that for every Banach spaces $E_1, \ldots, E_n$, and $p \geq 1$, the space of p-semi-integral forms $(\mathcal{L}_{si,p}(E_1, \ldots, E_n), \|\cdot\|_{si,p})$ is isometric to $(E_1 \otimes \cdots \otimes E_n \otimes \mathbb{K}, \widetilde{\sigma}_p)'$.

On the other hand, by a slight modification of the proof of Proposition 6, alternatively, we obtain the representation of the space of p-semi-integral forms as the dual of the tensor product endowed with the σ_p-norm.

7 Proposition. *Let $E_1, \ldots, E_n$ be Banach spaces, and let $p \geq 1$. Then $(\mathcal{L}_{si,p}(E_1, \ldots, E_n), \|\cdot\|_{si,p})$ is isometrically isomorphic to $(E_1 \otimes \cdots \otimes E_n, \sigma_p)'$ through the mapping $T \longrightarrow \phi_T$, where $\phi_T(x_1 \otimes \cdots \otimes x_n) = T(x_1, \ldots, x_n)$ for every $x_l \in E_l$, $l = 1, \ldots, n$.*

It is interesting to observe that $(E_1 \otimes \cdots \otimes E_n \otimes \mathbb{K}, \widetilde{\sigma}_p)'$ is not isometric to $(E_1 \otimes \cdots \otimes E_n, \widetilde{\sigma}_p)'$, but as a consequence of Propositions 6 and 7 we see that $(E_1 \otimes \cdots \otimes E_n \otimes \mathbb{K}, \widetilde{\sigma}_p)'$ is isometric to $(E_1 \otimes \cdots \otimes E_n, \sigma_p)'$.

Recall that a linear operator $u : E \longrightarrow F$ is said to be absolutely p-summing if $(u(x_j))_{j=1}^\infty \in l_p(F)$ whenever $(x_j)_{j=1}^\infty \in l_p^w(E)$. The vector space (operator ideal) composed by all absolutely p-summing operators from E to F is denoted by $\mathcal{L}_{as,p}(E; F)$. Hence the class of absolutely p-summing linear mappings coincides with the class of p-semi integral linear mappings. So in the linear case we prefer to write $\mathcal{L}_{as,p}(E; F)$ (resp. $\| \,.\, \|_{as,p}$) instead of $\mathcal{L}_{si,p}(E; F)$ (resp. $\| \,.\, \|_{si,p}$). For the theory of absolutely summing operators we refer to [3].

Below, inspired by a result of D. Pérez-García [12], we show that the norm σ_p is well behaved in connection with p-semi integral mappings.

8 Proposition. *Let $E_1, \ldots, E_n$ and F be Banach spaces and let $p \geq 1$. Then we have the following:*

(a) *If* $T : E_1 \otimes \cdots \otimes E_n \longrightarrow F$ *is a linear operator, then* $T \in \mathcal{L}((E_1 \widetilde{\otimes} \cdots \widetilde{\otimes} E_n, \sigma_p); F)$ *if and only if* $\varphi \circ T \in (E_1 \widetilde{\otimes} \cdots \widetilde{\otimes} E_n, \sigma_p)'$ *for every* $\varphi \in B_{F'}$. *In this case we have:*

$$\| T \|_{\mathcal{L}((\widetilde{E_1} \otimes \cdots \otimes \widetilde{E_n}, \sigma_p); F)} = \sup_{\varphi \in B_{F'}} \| \varphi \circ T \|_{(\widetilde{E_1} \otimes \cdots \otimes \widetilde{E_n}, \sigma_p)'} .$$

(b) *A multilinear mapping* $T : E_1 \times \cdots \times E_n \longrightarrow F$ *is p-semi integral if its associated linear mapping* $\widetilde{T} : E_1 \otimes \cdots \otimes E_n \longrightarrow F$, *given by* $\widetilde{T}(x_1 \otimes \cdots \otimes x_n) = T(x_1, \ldots, x_n)$ *for every* $x_l \in E_l$, $l = 1, \ldots, n$, *is* σ_p*-continuous and p-semi integral. In this case we have*

$$\| T \| \leq \| T \|_{si,p} \leq \| \widetilde{T} \|_{si,p} .$$

Conversely, if $T \in \mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$, *then the associated linear mapping* $\widetilde{T}$ *is* σ_p*-continuous, that is,* $\widetilde{T} \in \mathcal{L}((E_1 \otimes \cdots \otimes E_n, \sigma_p); F)$. *In this case we have:*

$$\| T \| \leq \| \widetilde{T} \|_{\mathcal{L}((E_1 \otimes \cdots \otimes E_n, \sigma_p); F)} \leq \| T \|_{si,p} .$$

Proof. (a) The non-trivial implication of the first assertion is an easy consequence of the closed graph theorem. To show the second assertion let $u_0 \in E_1 \otimes \cdots \otimes E_n$ with $Tu_0 \neq 0$. Then by the Hanh-Banach Theorem there exists a $\varphi_0 \in B_{F'}$ such that $\varphi_0(Tu_0) = \| Tu_0 \|$. Therefore for every $\varphi \in B_{F'}$ we have that

$$\| Tu_0 \| \leq \sup_{\varphi \in B_{F'}} | \varphi \circ T(u_0) | \leq \sup_{\varphi \in B_{F'}} \| \varphi \circ T \|_{(\widetilde{E_1} \otimes \cdots \otimes \widetilde{E_n}, \sigma_p)'} \sigma_p(u_0),$$

which shows that

$$\| T \|_{\mathcal{L}((\widetilde{E_1} \otimes \cdots \otimes \widetilde{E_n}, \sigma_p); F)} \leq \sup_{\varphi \in B_{F'}} \| \varphi \circ T \|_{(\widetilde{E_1} \otimes \cdots \otimes \widetilde{E_n}, \sigma_p)'} .$$

Since the reverse inequality is immediate we have (a).

(b) Suppose $\widetilde{T} \in \mathcal{L}_{si,p}((E_1 \otimes \cdots \otimes E_n, \sigma_p); F)$. Then by Proposition 7 and

Theorem 1 it follows that

$$\begin{aligned}
&\left(\sum_{j=1}^{m}\|T(x_{1,j},\ldots,x_{n,j})\|^p\right)^{1/p}\\
&\leq\| \widetilde{T} \|_{si,p}\left(\sup_{\varphi\in B_{(E_1\otimes\cdots\otimes E_n,\sigma_p)'}}\sum_{j=1}^{m}|\varphi(x_{1,j}\otimes\cdots\otimes x_{n,j})|^p\right)^{1/p}\\
&=\| \widetilde{T} \|_{si,p}\left(\sup_{S\in B_{(\mathcal{L}_{si,p}(E_1,\cdots,E_n),\|\cdot\|_{si,p})}}\sum_{j=1}^{m}|S(x_{1,j},\cdots,x_{n,j})|^p\right)^{1/p}\\
&\leq\| \widetilde{T} \|_{si,p}\left(\sup_{\substack{\varphi_l\in B_{E_l'}\\ l=1,\ldots,n}}\sum_{j=1}^{m}|\varphi_1(x_{1,j})\cdots\varphi_n(x_{n,j})|^p\right)^{1/p},
\end{aligned}$$

which shows that $T\in\mathcal{L}_{si,p}(E_1,\ldots,E_n;F)$ with $\| T \|_{si,p}\leq\| \widetilde{T} \|_{si,p}$. The fact that $\| T \|\leq\| T \|_{si,p}$ follows easily from Theorem 1.

To show the converse, suppose now $T\in\mathcal{L}_{si,p}(E_1,\ldots,E_n;F)$, and let $u\in E_1\otimes\cdots\otimes E_n$. Choosing a representation $u=\sum_{j=1}^{m}\lambda_j x_{1,j}\otimes\cdots\otimes x_{n,j}$, from the Hölder's inequality and Theorem 1 it follows that

$$\begin{aligned}
\|\widetilde{T}(u)\|^p &\leq \|(\lambda_j)_{j=1}^m\|_q^p\sum_{j=1}^{m}\| T(x_{1,j},\ldots,x_{n,j}) \|^p\\
&\leq \|(\lambda_j)_{j=1}^m\|_q^p\|T\|_{si,p}^p\sup_{\substack{\varphi_l\in B_{E_l'}\\ l=1,\ldots,n}}\sum_{j=1}^{m}|\varphi_1(x_{1,j})\cdots\varphi_n(x_{n,j})|^p.
\end{aligned}$$

Hence $\|\widetilde{T}(u)\|\leq\|T\|_{si,p}\sigma_p(u)$, and so $\widetilde{T}$ is σ_p-continuous with $\| \widetilde{T} \|_{\mathcal{L}((E_1\otimes\cdots\otimes E_n,\sigma_p);F)}\leq\| T \|_{si,p}$. Finally, since σ_p is a reasonable cross-norm, it readily follows that $\| T \|\leq\| \widetilde{T} \|_{\mathcal{L}((E_1\otimes\cdots\otimes E_n,\sigma_p);F)}$, which completes the proof of (b).

$\boxed{QED}$

Proposition 8(b) can be seen as a weak vector-valued version of Proposition 7. We do not know if, in general, $\widetilde{T}\in\mathcal{L}_{si,p}((E_1\otimes\cdots\otimes E_n,\sigma_p);F)$ whenever $T\in\mathcal{L}_{si,p}(E_1,\ldots,E_n;F)$.

We end this section by giving another property of the p-semi integral multilinear mappings.

9 Proposition. *[11, Teorema 5.1.14] If $T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$ then, for each $i = 1, \dots, n$, the mapping $T_i \colon E_i \longrightarrow \mathcal{L}(E_1, \overset{[i]}{\dots}, E_n; F)$, defined by $T_i(x_i)(x_1, \overset{[i]}{\dots}, x_n) := T(x_1, \dots, x_n)$, is absolutely p-summing with $T_i(x_i) \in \mathcal{L}_{si,p}(E_1, \overset{[i]}{\dots}, E_n; F)$. Furthermore,*

$$\| T \| = \| T_i \| \leq \|T_i\|_{as,p} \leq \|T\|_{si,p}.$$

Proof. A close examination of the proof of [11, Teorema 5.1.14] gives the first part. Since it is readily seen that $\| T \| = \| T_i \|$ and, it follows by Proposition 8(b) that $\| T_i \| \leq \| T_i \|_{si,p}$, we have the proof. $\boxed{QED}$

2 Tensor Product of Operators

In this section we consider the tensor product of linear operators in connection with the reasonable crossnorm $\widetilde{\sigma}_p$ (and σ_p). We show that the reasonable crossnorms $\widetilde{\sigma}_p$ and σ_p are actually tensor norms. The results of this section are similar to those ones given for the projective tensor product in connection with bilinear mappings in [14] with the same patterns in corresponding proofs [see [14, Propositions 2.3 and 2.4]].

In what follows we use the notation $\widetilde{\sigma}_{p;E_1,\dots,E_n}$ to emphasize that the crossnorm $\widetilde{\sigma}_p$ is considered on $E_1 \otimes \cdots \otimes E_n$.

10 Proposition. *Let $T_i \in \mathcal{L}(E_i; F_i)$, $i = 1, \dots, n$, $T \in \mathcal{L}(E; F)$ and $p \geq 1$. Then there is a unique continuous linear operator $T_1 \otimes_{\widetilde{\sigma}_p} \cdots \otimes_{\widetilde{\sigma}_p} T_n \otimes_{\widetilde{\sigma}_p} T \colon (E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \tilde{\otimes} E, \widetilde{\sigma}_p) \longrightarrow (F_1 \tilde{\otimes} \cdots \tilde{\otimes} F_n \tilde{\otimes} F, \widetilde{\sigma}_p)$ such that*

$$T_1 \otimes_{\widetilde{\sigma}_p} \cdots \otimes_{\widetilde{\sigma}_p} T_n \otimes_{\widetilde{\sigma}_p} T(x_1 \otimes \cdots \otimes x_n \otimes x) = (T_1 x_1) \otimes \cdots \otimes (T_n x_n) \otimes (Tx)$$

for every $x_i \in E_i$, $i = 1, \dots, n$, and $x \in E$. Moreover

$$\|T_1 \otimes_{\widetilde{\sigma}_p} \cdots \otimes_{\widetilde{\sigma}_p} T_n \otimes_{\widetilde{\sigma}_p} T\| = \|T_1 \otimes \cdots \otimes T_n \otimes T\| = \|T_1\| \cdots \|T_n\| \|T\|.$$

Proof. Given linear operators $T_i \in \mathcal{L}(E_i; F_i)$, $i = 1, \dots, n$, and $T \in \mathcal{L}(E; F)$, there is a unique linear operator $T_1 \otimes \cdots \otimes T_n \otimes T \colon E_1 \otimes \cdots \otimes E_n \otimes E \longrightarrow F_1 \otimes \cdots \otimes F_n \otimes F$ such that

$$T_1 \otimes \cdots \otimes T_n \otimes T(x_1 \otimes \cdots \otimes x_n \otimes x) = (T_1 x_1) \otimes \cdots \otimes (T_n x_n) \otimes (Tx)$$

for every $x_i \in E_i$, $i = 1, \dots, n$ and $x \in E$ [see [14, p. 7]]. We may suppose $T_i \neq 0$, $i = 1, \dots, n$ and $T \neq 0$. Let $u \in E_1 \otimes \cdots \otimes E_n \otimes E$ and let $\sum_{j=1}^{m} \lambda_j x_{1,j} \otimes \cdots \otimes x_{n,j} \otimes x_j$ be a representation of u. Hence the sum

$$\sum_{j=1}^{m} \lambda_j T_1(x_{1,j}) \otimes \cdots \otimes T_n(x_{n,j}) \otimes T(x_j)$$

is a representation of $T_1 \otimes \cdots \otimes T_n \otimes T(u)$ in $F_1 \otimes \cdots \otimes F_n \otimes F$. Then, for every $p \geq 1$

$$\begin{aligned}
&\widetilde{\sigma}_{p;F_1,\ldots,F_n,F}(T_1 \otimes \cdots \otimes T_n \otimes T(u)) \\
&\quad \leq \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\phi_l \in B_{F_l'} \\ l=1,\ldots,n}} \sum_{j=1}^m |\phi_1(T_1 x_{1,j}) \ldots \phi_n(T_n x_{n,j})|^p \right)^{1/p} \|(Tx_j)_{j=1}^m\|_\infty \\
&\leq \|T_1\| \cdots \|T_n\| \|T\| \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\phi_l \in B_{E_l'} \\ l=1,\ldots,n}} \sum_{j=1}^m |\phi_1(x_{1,j}) \ldots \phi_n(x_{n,j})|^p \right)^{1/p} \|(x_j)_{j=1}^m\|_\infty
\end{aligned}$$

and we have that

$$\widetilde{\sigma}_{p;F_1,\ldots,F_n,F}(T_1 \otimes \cdots \otimes T_n \otimes T(u)) \leq \|T_1\| \cdots \|T_n\| \|T\| \widetilde{\sigma}_{p;E_1,\ldots,E_n,E}(u),$$

so that the linear operator $T_1 \otimes \cdots \otimes T_n \otimes T$ is continuous for the crossnorms on $E_1 \otimes \cdots \otimes E_n \otimes E$ and $F_1 \otimes \cdots \otimes F_n \otimes F$ and $\|T_1 \otimes \cdots \otimes T_n \otimes T\| \leq \|T_1\| \cdots \|T_n\| \|T\|$. On the other hand, as $\widetilde{\sigma}_p$ is an reasonable crossnorm we get that

$$\begin{aligned}
\|T_1(x_1)\| \cdots \|T_n(x_n)\| \|T(x)\| &= \widetilde{\sigma}_{p;F_1,\ldots,F_n,F}(T_1(x_1) \otimes \cdots \otimes T_n(x_n) \otimes T(x)) \\
&\leq \|T_1 \otimes \cdots \otimes T_n \otimes T\| \quad \widetilde{\sigma}_{p;E_1,\ldots,E_n,E}(x_1 \otimes \cdots \otimes x_n \otimes x) \\
&\qquad = \|T_1 \otimes \cdots \otimes T_n \otimes T\| \|x_1\| \cdots \|x_n\| \|x\|,
\end{aligned}$$

[see [14, Proposition 6.1]], and therefore $\|T_1 \otimes \cdots \otimes T_n \otimes T\| \geq \|T_1\| \cdots \|T_n\| \|T\|$. Hence we have that

$$\|T_1 \otimes \cdots \otimes T_n \otimes T\| = \|T_1\| \cdots \|T_n\| \|T\|$$

Now taking the unique continuous extension of the operator $T_1 \otimes \cdots \otimes T_n \otimes T$ to the completions of $(E_1 \otimes \cdots \otimes E_n \otimes E, \widetilde{\sigma}_p)$ and $(F_1 \otimes \cdots \otimes F_n \otimes F, \widetilde{\sigma}_p)$, which we denote by $T_1 \otimes_{\widetilde{\sigma}_p} \cdots \otimes_{\widetilde{\sigma}_p} T_n \otimes_{\widetilde{\sigma}_p} T$, we obtain a unique linear operator from $(E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n \tilde{\otimes} E, \widetilde{\sigma}_p)$ into $(F_1 \tilde{\otimes} \cdots \tilde{\otimes} F_n \tilde{\otimes} F, \widetilde{\sigma}_p)$ with the norm $\|T_1 \otimes_{\widetilde{\sigma}_p} \cdots \otimes_{\widetilde{\sigma}_p} T_n \otimes_{\widetilde{\sigma}_p} T\| = \|T_1\| \cdots \|T_n\| \|T\|$. $\boxed{QED}$

The $\widetilde{\sigma}_p$-tensor product does not respect subspaces but respects 1-complemented subspaces. Indeed; if E_0 is a subspace of E, then $E_0 \otimes F$ is an algebraic subspace of $E \otimes F$, but the norm induced on $E_0 \otimes F$ by $(E \otimes F, \widetilde{\sigma}_p)$ is not, in general the $\widetilde{\sigma}_p$ norm on $E_0 \otimes F$. In fact, if we take $u \in E_0 \otimes F$, then we see that

$$\begin{aligned}\widetilde{\sigma}_{p;E,F}(u) &= \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\varphi \in B_{E'}} \sum_{j=1}^m |\varphi(x_j)|^p \right)^{1/p} \|(y_j)_{j=1}^m\|_\infty \\ &\leq \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\psi \in B_{E_0'}} \sum_{j=1}^m |\psi(x_j)|^p \right)^{1/p} \|(y_j)_{j=1}^m\|_\infty = \widetilde{\sigma}_{p;E_0,F}(u)\end{aligned}$$

since the set of representations of u become bigger when we enlarge the space E_0 to E. Similarly if F_0 is a subspace of F, then $E \otimes F_0$ is an algebraic subspace of $E \otimes F$, but the norm induced on $E \otimes F_0$ by $(E \otimes F, \widetilde{\sigma}_p)$ is not, in general the $\widetilde{\sigma}_p$ norm on $E \otimes F_0$. Whereas for complemented subspaces we have:

11 Proposition. *Let $M_1, \dots, M_n$, N be complemented subspaces of $E_1, \dots, E_n$, F respectively. Then $M_1 \otimes \cdots \otimes M_n \otimes N$ is complemented in $(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)$ and the norm on $M_1 \otimes \cdots \otimes M_n \otimes N$ induced by $\widetilde{\sigma}_{p;E_1,\dots,E_n,F}$ is equivalent to $\widetilde{\sigma}_{p;M_1,\dots,M_n,N}$. Moreover, if $M_1, \dots, M_n$ and N are 1-complemented, then $(M_1 \otimes \cdots \otimes M_n \otimes N, \widetilde{\sigma}_p)$ is 1-complemented in $(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)$ as well.*

Proof. Let $P_1, \dots, P_n$, Q be projections from $E_1, \dots, E_n$, F onto $M_1, \dots, M_n$, N respectively. One can easily show that $P_1 \otimes \cdots \otimes P_n \otimes Q$ is a projection of $(E_1 \otimes \cdots \otimes E_n \otimes F, \widetilde{\sigma}_p)$ onto $M_1 \otimes \cdots \otimes M_n \otimes N$. We just have proved above that $\widetilde{\sigma}_{p;E,F}(u) \leq \widetilde{\sigma}_{p;M,N}(u)$ for $u \in M \otimes N$, and the same argument shows that $\widetilde{\sigma}_{p;E_1,\dots,E_n,F}(u) \leq \widetilde{\sigma}_{p;M_1,\dots,M_n,N}(u)$ for $u \in M_1 \otimes \cdots \otimes M_n \otimes N$.

Let $u \in M_1 \otimes \cdots \otimes M_n \otimes N$ and let $\sum_{j=1}^m \lambda_j x_{1,j} \cdots \otimes x_{n,j} \otimes y_j$ be a representation of u in $E_1 \otimes \cdots \otimes E_n \otimes F$. Then $u = P_1 \otimes \cdots \otimes P_n \otimes Q(u) = \sum_{j=1}^m \lambda_j P_1(x_{1,j}) \otimes \cdots \otimes P_n(x_{n,j}) \otimes Q(y_j)$ is a representation of u in $M_1 \otimes \cdots \otimes M_n \otimes N$. Therefore, by the argument used in the proof of Proposition 10 we obtain

$$\begin{aligned}&\widetilde{\sigma}_{p;M_1,\dots,M_n,N}(u) \\ &\leq \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\phi_l \in B_{M_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\phi_1(P_1(x_{1,j})) \cdots \phi_n(P_n(x_{n,j}))|^p \right)^{1/p} \|(Q(y_j))_{j=1}^m\|_\infty \\ &\leq \|P_1\| \cdots \|P_n\| \|Q\| \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\phi_l \in B_{E_l'} \\ l=1,\dots,n}} \sum_{j=1}^m |\phi_1(x_{1,j}) \cdots \phi_n(x_{n,j})|^p \right)^{1/p} \|(y_j)_{j=1}^m\|_\infty .\end{aligned}$$

Since this holds for every representation of u in $E_1 \otimes \cdots \otimes E_n \otimes F$, it follows that

$$\widetilde{\sigma}_{p;E_1,\ldots,E_n,F}(u) \leq \widetilde{\sigma}_{p;M_1,\ldots,M_n,N}(u) \leq \|P_1\| \cdots \|P_n\| \|Q\| \widetilde{\sigma}_{p;E_1,\ldots,E_n,F}(u).$$

Now, if $M_1, \ldots, M_n$ and N are complemented by projections of norm one, then we have that $\widetilde{\sigma}_{p;E_1,\ldots,E_n,F}(u) = \widetilde{\sigma}_{p;M_1,\ldots,M_n,N}(u)$ for every $u \in M_1 \otimes \cdots \otimes M_n \otimes N$, and by Proposition 10 it follows that $\|P_1 \otimes \cdots \otimes P_n \otimes Q\| = \|P_1\| \cdots \|P_1\| \|Q\| = 1$, as we desired. $\boxed{QED}$

We note that an analogous result to Proposition 10, in a similar way, can be obtained for σ_p also. As well, like the case of $\widetilde{\sigma}_p$, and with analogous reasonings, the σ_p-tensor product does not respect subspaces but respects 1-complemented subspaces.

3 Connection with multilinear mappings of finite type

We recall that a multilinear mapping $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to be of finite type if it has a finite representation of the form

$$T = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \times \cdots \times \varphi_{n,j} b_j \tag{2}$$

where $\lambda_j \in \mathbb{K}$, $\varphi_{l,j} \in E_l'$, $l = 1, \ldots, n$, $b_j \in F$, $j = 1, \ldots, m$. We denote by $\mathcal{L}_f(E_1, \ldots, E_n; F)$ the vector subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ of all n-linear mappings of finite type. It is plain that multilinear mappings of finite type are p-semi-integral, that is, $\mathcal{L}_f(E_1, \ldots, E_n; F) \subset \mathcal{L}_{si,p}(E_1, \ldots, E_n; F)$. It is clear that to each operator in $\mathcal{L}_f(E_1, \ldots, E_n; F)$ corresponds a tensor in $E_1' \otimes \cdots \otimes E_n' \otimes F$ via the canonical mapping

$$u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \otimes \cdots \otimes \varphi_{n,j} \otimes b_j \longrightarrow T_u = \sum_{j=1}^{m} \lambda_j \varphi_{1,j} \times \cdots \times \varphi_{n,j} b_j, \tag{3}$$

where $\lambda_j \in \mathbb{K}$, $\varphi_{l,j} \in E_l'$, $l = 1, \ldots, n$, $b_j \in F$, $j = 1, \ldots, m$. Next we will see that, in some cases, these mappings are isometries.

12 Proposition. *Let $E_1, \ldots, E_n$ and F be Banach spaces and let $p \geq 1$. Given $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$, define*

$$\|T\|_{f,p} := \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{\phi_l \in B_{E_l''} \\ l=1,\ldots,n}} \sum_{j=1}^{m} |\phi_1(\varphi_{1,j}) \cdots \phi_n(\varphi_{n,j})|^p \right)^{1/p} \|(b_j)_{j=1}^m\|_\infty$$

where the infimum is taken over all representations of T as in (2), and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Then $\|\cdot\|_{f,p}$ is a norm on $\mathcal{L}_f(E_1,\ldots,E_n;F)$ with the following properties :

(a) For every $u \in E_1^{'} \otimes \cdots \otimes E_n^{'} \otimes F$ we have that $\|T_u\| \leq \|T_u\|_{f,p} = \widetilde{\sigma}_p(u)$.

Consequently, $(\mathcal{L}_f(E_1,\ldots,E_n;F), \|\,.\,\|_{f,p})$ is isometrically isomorphic to $(E_1^{'} \otimes \cdots \otimes E_n^{'} \otimes F, \widetilde{\sigma}_p)$ via the mapping given in (3).

(b) For every $\varphi_l \in E_l'$, $l=1,\ldots,n$, and $b \in F$ we have that $\|\varphi_1 \times \cdots \times \varphi_n b\|_{f,p} = \|\varphi_1\| \cdots \|\varphi_n\| \cdot \|b\|$.

Proof. Following the lines of the proof of Proposition 2 it is easy to see that $\|\cdot\|_{f,p}$ is a norm on $\mathcal{L}_f(E_1,\ldots,E_n;F)$.

(a) Since the equality $\|T_u\|_{f,p} = \widetilde{\sigma}_p(u)$ is trivial we show that $\|T_u\| \leq \|T_u\|_{f,p}$. Given $x_l \in E_l$ with $x_l \neq 0$, $l=1,\ldots,n$, by Hölder's inequality we have

$$\begin{aligned}&\|T_u(x_1,\ldots,x_n)\|^p \\ &\leq \|x_1\|^p \cdots \|x_n\|^p \|(b_j)_{j=1}^m\|_\infty^p \|(\lambda_j)_{j=1}^m\|_q^p \sup_{\substack{\phi_l \in B_{E_l''} \\ l=1,\ldots,n}} \sum_{j=1}^m |\phi_1(\varphi_{1,j}) \cdots \phi_n(\varphi_{n,j})|^p.\end{aligned}$$

So, it follows that $\|T_u(x_1,\ldots,x_n)\| \leq \|T_u\|_{f,p}\|x_1\| \cdots \|x_n\|$ and we have (a).

(b) Take $\varphi_l \in E_l'$, $l=1,\ldots,n$, and $b \in F$. It is immediate that $\|\varphi_1 \times \cdots \times \varphi_n b\|_{f,p} \leq \|\varphi_1\| \cdots \|\varphi_n\| \cdot \|b\|$. To prove the reverse inequality we use (a). For every $x_l \in E_l$, $l=1,\ldots,n$, we have

$$\begin{aligned}|\varphi_1(x_1)| \cdots |\varphi_n(x_n)| \|b\| &\leq \|\varphi_1 \times \cdots \times \varphi_n b\| \|x_1\| \cdots \|x_n\| \\ &\leq \|\varphi_1 \times \cdots \times \varphi_n b\|_{f,p} \|x_1\| \cdots \|x_n\|.\end{aligned}$$

Taking the supremum over B_{E_l}, $l=1,\ldots,n$, we see that $\|\varphi_1\| \cdots \|\varphi_n\| \cdot \|b\| \leq \|\varphi_1 \times \cdots \times \varphi_n b\|_{f,p}$.

$\boxed{QED}$

By Proposition 12(b) we see that $\|\varphi_1 \times \cdots \times \varphi_n b\|_{f,p} = \|\varphi_1 \times \cdots \times \varphi_n b\|_{si,p}$ for every $\varphi_l \in E_l'$, $l=1,\ldots,n$, and every $b \in F$ with $p \geq 1$. We do not know if $\|T\|_{f,p} = \|T\|_{si,p}$ whenever $T \in L_f(E_1,\ldots,E_n;F)$.

13 Remark. When $E_1, \ldots, E_n$ are reflexive Banach spaces the norm $\|\cdot\|_{f,p}$ on $\mathcal{L}_f(E_1, \ldots, E_n; F)$ reduces to the following equivalent formulation: Given $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$, we have that

$$\|T\|_{f,p} = \inf \|(\lambda_j)_{j=1}^m\|_q \left(\sup_{\substack{x_l \in B_{E_l} \\ l=1,\ldots,n}} \sum_{j=1}^m |\varphi_{1,j}(x_1) \cdots \varphi_{n,j}(x_n)|^p \right)^{1/p} \|(b_j)_{j=1}^m\|_\infty$$

where the infimum is taken over all representations of T as in (2), and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Next result provides a relation between $(\mathcal{L}_{si,p}(E'_1, \ldots, E'_n; F'), \|\cdot\|_{si,p})$ and $(\mathcal{L}_f(E_1, \ldots, E_n; F), \|\cdot\|_{f,p})$, which gives a predual of $(\mathcal{L}_{si,p}(E'_1, \ldots, E'_n; F'), \|\cdot\|_{si,p})$, and also shows another predual of $(\mathcal{L}_{si,p}(E_1, \ldots, E_n; F'), \|\cdot\|_{si,p})$ in case of $E_1, \ldots, E_n$ being reflexive spaces.

14 Proposition. *Let $E_1, \ldots, E_n$ be Banach spaces and let $p \geq 1$.*

(a) Then $(\mathcal{L}_{si,p}(E'_1, \ldots, E'_n; F'), \|\cdot\|_{si,p})$ is isometrically isomorphic to $(\mathcal{L}_f(E_1, \ldots, E_n; F), \|\cdot\|_{f,p})'$ by the mapping

$$T(\psi)(\varphi_1, \ldots \varphi_n)(b) = \psi(\varphi_1 \times \cdots \times \varphi_n b),$$

where $b \in F$, $\varphi_l \in E'_l$, $l = 1, \ldots, n$, and $\psi \in (L_f(E_1, \ldots, E_n; F), \|\cdot\|_{f,p})'$.

If, in addition, $E_1, \ldots, E_n$ are reflexive Banach spaces then

(b) $(\mathcal{L}_{si,p}(E_1, \ldots, E_n; F'), \|\cdot\|_{si,p})$ and $(\mathcal{L}_f(E'_1, \ldots, E'_n; F), \|\cdot\|_{f,p})'$ are isometric via the mapping

$$T(\psi)(x_1, \ldots x_n)(b) = \psi(x_1 \times \cdots \times x_n b),$$

where $b \in F$, $x_l \in E_l$, $l = 1, \ldots, n$, and $\psi \in (\mathcal{L}_f(E'_1, \ldots, E'_n; F), \|\cdot\|_{f,p})'$.

Proof. (a) follows from Propositions 6 and 12 and (b) is a straightforward consequence of (a) $\boxed{QED}$

In the next by combining the previous results and taking $F = \mathbb{K}$, in particular, we obtain the following.

15 Corollary. *Let $E_1, \ldots, E_n$ be Banach spaces and let $p \geq 1$. Then the following isometries hold true:*

(a) $(\mathcal{L}_{si,p}(E'_1, \ldots, E'_n), \|\cdot\|_{si,p}) \cong (E'_1 \otimes \cdots \otimes E'_n; \sigma_p)' \cong (E'_1 \otimes \cdots \otimes E'_n \otimes \mathbb{K}; \widetilde{\sigma}_p)' \cong (\mathcal{L}_f(E_1, \ldots, E_n), \|\cdot\|_{f,p})'$.

If, in addition, $E_1, \ldots, E_n$ are reflexive Banach spaces then the following isometries hold true:

(b) $(\mathcal{L}_{si,p}(E_1,\ldots,E_n),\|\cdot\|_{si,p}) \cong (E_1\otimes\cdots\otimes E_n;\sigma_p)' \cong (E_1\otimes\cdots\otimes E_n\otimes\mathbb{K};\widetilde{\sigma}_p)' \cong (\mathcal{L}_f(E_1',\ldots,E_n'),\|\cdot\|_{f,p})'$.

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